

A First Study of the Effects of a non-Sinusoidal Motor

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Throughout this analysis, please refer to Figure 1 for voltage polarities, sign conventions, etc. The self-inductances of the windings are assumed to be negligible for this analysis.

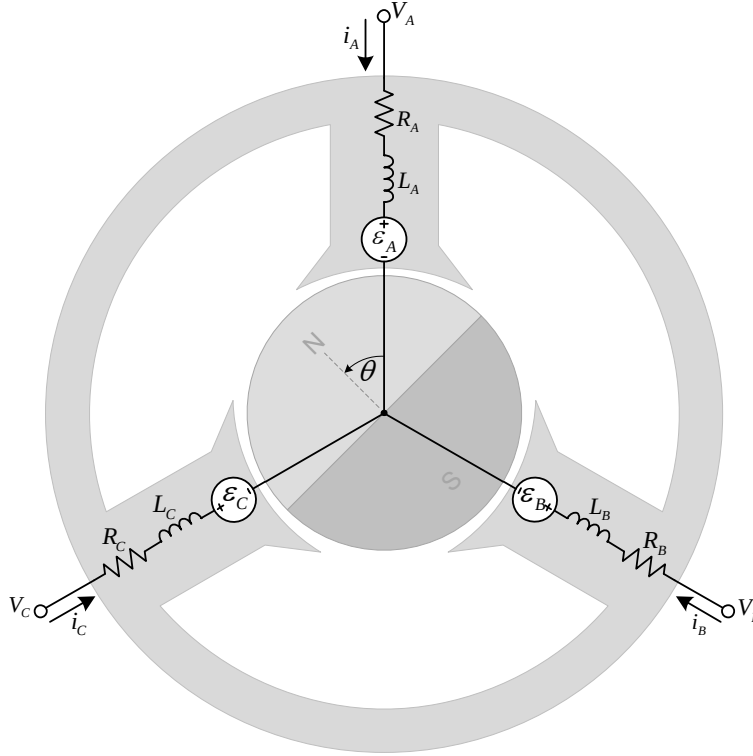


Figure 1. A diagram of a brushless DC motor

Formulation of non-sinusoidal winding EMF's and torques

Many times when analyzing brushless DC motors, we assume many things about the motor. For example, we'll often begin assuming the motor winding EMF's and torque profiles are given by:

$$\varepsilon_A = k_v \dot{\theta} \sin(\theta) \quad [1.1]$$

$$\varepsilon_B = k_v \dot{\theta} \sin(\theta + 120^\circ) \quad [1.2]$$

$$\varepsilon_C = k_v \dot{\theta} \sin(\theta + 240^\circ) \quad [1.3]$$

and:

$$T_A = k_t i_A \sin(\theta) \quad [1.4]$$

$$T_B = k_t i_B \sin(\theta + 120^\circ) \quad [1.5]$$

$$T_C = k_t i_C \sin(\theta + 240^\circ) \quad [1.6]$$

The above equations assume the motor is perfectly symmetric, and is "sinusoidal." But let's assume the motor winding EMF's and torque profiles are not sinusoidal. A much more general way of expressing them is as follows:

$$\varepsilon_A = k_v \dot{\theta} \cdot p(\theta) \quad [1.7]$$

$$\varepsilon_B = k_v \dot{\theta} \cdot p(\theta + 120^\circ) \quad [1.8]$$

$$\varepsilon_C = k_v \dot{\theta} \cdot p(\theta + 240^\circ) \quad [1.9]$$

and

$$T_A = k_t i_A \cdot p(\theta) \quad [1.10]$$

$$T_B = k_t i_B \cdot p(\theta + 120^\circ) \quad [1.11]$$

$$T_C = k_t i_C \cdot p(\theta + 240^\circ) \quad [1.12]$$

where $p(\theta)$ is a periodic function of period 2π and with zero average. Note that we are still enforcing a type of symmetry by assuming each of the three windings are 120 degrees apart. However, the shape of the curves is arbitrary, as long as they have a period of 2π .

We can express $p(\theta)$ in terms of its Fourier series:

$$p(\theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) \quad [1.13]$$

Let's now add the following two constraints on $p(\theta)$:

$$p(\theta) = -p(-\theta) \quad [1.14]$$

$$p(\theta) = p(\pi - \theta) \quad [1.15]$$

The first of these constrains $p(\theta)$ to be an odd function (hopefully a familiar concept).

The second constraint imposes a sort of symmetry on $p(\theta)$. Specifically, it states that the vertical line at $\theta = \frac{\pi}{2}$ is a line of symmetry.

First we plug [1.14] into [1.13]:

$$\sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) = -\sum_{n=1}^{\infty} a_n \sin(-n\theta) + b_n \cos(-n\theta) \quad [1.16]$$

Combining the summations onto the left side:

$$\sum_{n=1}^{\infty} a_n [\sin(n\theta) + \sin(-n\theta)] + b_n [\cos(n\theta) + \cos(-n\theta)] = 0 \quad [1.17]$$

Recall that due to the evenness and oddness of cosine and sine, respectively:

$$\cos(-x) = \cos(x)$$

and

$$\sin(-x) = -\sin(x)$$

These relations simplify [1.17] to:

$$2 \sum_{n=1}^{\infty} b_n \cos(n\theta) = 0 \quad [1.18]$$

Since the family of $\cos(n\theta)$ functions is linearly independent, the only way to satisfy [1.18] for all θ is for each of the b_n coefficients to be zero:

$$b_n = 0 \quad ; \quad n = 1, 2, 3, \dots, \infty$$

This means we can express $p(\theta)$ as just a sum of sine terms:

$$p(\theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta) \quad [1.19]$$

And now we use the constraint in [1.15]. Plugging it into [1.19], we get:

$$\begin{aligned} \sum_{n=1}^{\infty} a_n \sin(n\theta) &= \sum_{n=1}^{\infty} a_n \sin(n(\pi - \theta)) \\ &= \sum_{n=1}^{\infty} a_n \sin(n\pi - n\theta) \\ &= -\sum_{n=1}^{\infty} a_n \sin(n\theta - n\pi) \end{aligned} \quad [1.20]$$

where we've again used the oddness of sine here. Again combining terms:

$$\sum_{n=1}^{\infty} a_n [\sin(n\theta) + \sin(n\theta - n\pi)] = 0 \quad [1.21]$$

Noting that:

$$\sin(x - n\pi) = \begin{cases} \sin(x) & ; \text{ even } n \\ -\sin(x) & ; \text{ odd } n \end{cases} \quad [1.22]$$

This simplifies [1.21] to:

$$2 \sum_{n=2,4,\dots}^{\infty} a_n \sin(n\theta) = 0 \quad [1.23]$$

With the same reasoning as above, this implies all the even sine coefficients are zero:

$$a_n = 0 \quad ; \quad n = 2, 4, 6, \dots, \infty \quad [1.24]$$

This means we can express $p(\theta)$ as just:

$$p(\theta) = \sum_{n=1,3,5,\dots}^{\infty} a_n \sin(n\theta) \quad [1.25]$$

Finally, we'll add a third, somewhat arbitrary constraint that essentially normalizes our function $p(\theta)$.

We constrain the amplitude of the first component (the fundamental component) to be unity:

$$a_1 = 1 \quad [1.26]$$

This simplifies [1.25] to:

$$p(\theta) = \sin(\theta) + \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta) \quad [1.27]$$

This last constraint doesn't have a much of a physical interpretation beyond the obvious. The maximum value of $p(\theta)$, for example, can still be more than or less than one.

Derivation of Average Motor Torque using Trapezoidal Commutation

The total torque produced by the motor is given by:

$$\begin{aligned} T &= T_A + T_B + T_C \\ &= k_t [i_A \cdot p(\theta) + i_B \cdot p(\theta + 120^\circ) + i_C \cdot p(\theta + 240^\circ)] \end{aligned} \quad [1.28]$$

In the case of trapezoidal commutation, the winding currents are made by the motor controller to be:

$$i_A = \begin{cases} +i_0 & ; \quad 30^0 \leq \theta < 150^0 \\ 0 & ; \quad 150^0 \leq \theta < 210^0, 330^0 \leq \theta < 390^0 \\ -i_0 & ; \quad 210^0 \leq \theta < 330^0 \end{cases} \quad [1.29]$$

$$i_B = \begin{cases} +i_0 & ; \quad -90^0 \leq \theta < 30^0 \\ 0 & ; \quad 30^0 \leq \theta < 90^0, 210^0 \leq \theta < 270^0 \\ -i_0 & ; \quad 90^0 \leq \theta < 210^0 \end{cases} \quad [1.30]$$

$$i_C = \begin{cases} +i_0 & ; \quad 150^0 \leq \theta < 270^0 \\ 0 & ; \quad 90^0 \leq \theta < 150^0, 270^0 \leq \theta < 330^0 \\ -i_0 & ; \quad -30^0 \leq \theta < 90^0 \end{cases} \quad [1.31]$$

The average torque produced over one of the six commutation states is:

$$\hat{T} = \frac{3}{\pi} \int_{30^0}^{90^0} k_t i_0 \cdot p(\theta) - k_t i_0 \cdot p(\theta + 240^0) d\theta = \frac{3k_t i_0}{\pi} \int_{30^0}^{90^0} p(\theta) - p(\theta + 240^0) d\theta \quad [1.32]$$

Substituting equation [1.25] (not [1.27]) in for $p(\theta)$, we get:

$$\begin{aligned} \hat{T} &= \frac{3k_t i_0}{\pi} \int_{30^0}^{90^0} \left[\sum_{n=1,3,5,\dots}^{\infty} a_n \sin(n\theta) - \sum_{n=1,3,5,\dots}^{\infty} a_n \sin\left(n\theta + n\frac{4\pi}{3}\right) \right] d\theta \\ &= \frac{3k_t i_0}{\pi} \left[- \sum_{n=1,3,5,\dots}^{\infty} \frac{a_n}{n} \cos(n\theta) + \sum_{n=1,3,5,\dots}^{\infty} \frac{a_n}{n} \cos\left(n\theta + n\frac{4\pi}{3}\right) \right] \Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\ &= \frac{3k_t i_0}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{a_n}{n} \left[\cos\left(n\theta + n\frac{4\pi}{3}\right) - \cos(n\theta) \right] \Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \end{aligned} \quad [1.33]$$

This evaluates to (note the pattern):

$$\begin{aligned} \hat{T} &= \frac{3k_t i_0}{\pi} \left(\sqrt{3}a_1 - \frac{\sqrt{3}}{5}a_5 - \frac{\sqrt{3}}{7}a_7 + \frac{\sqrt{3}}{11}a_{11} + \frac{\sqrt{3}}{13}a_{13} - \frac{\sqrt{3}}{17}a_{17} - \frac{\sqrt{3}}{19}a_{19} + \dots \right) \\ &= \frac{3\sqrt{3}k_t i_0}{\pi} \left(a_1 - \frac{a_5}{5} - \frac{a_7}{7} + \frac{a_{11}}{11} + \frac{a_{13}}{13} - \frac{a_{17}}{17} - \frac{a_{19}}{19} + \dots \right) \end{aligned} \quad [1.34]$$

Note the absence of terms for $n=3, 9, 15$, etc. These terms are cancelled due to the $n\frac{4\pi}{3}$ quantity in

the first cosine term in [1.32] being an integer multiple of 2π . Assuming our normalizing constraint (equation [1.26]) is satisfied, we arrive at the final form of the total motor torque:

$$\boxed{\hat{T} = \frac{3\sqrt{3}k_t i_0}{\pi} \left(1 - \frac{a_5}{5} - \frac{a_7}{7} + \frac{a_{11}}{11} + \frac{a_{13}}{13} - \frac{a_{17}}{17} - \frac{a_{19}}{19} + \dots \right)} \quad [1.35]$$

A good sanity check is to consider the case of a sinusoidal motor, in which case all Fourier coefficients above the fundamental component are zero. In this case, [1.35] simplifies to:

$$\hat{T}_{\text{sine_motor}} = \frac{3\sqrt{3}k_t i_0}{\pi} \quad [1.36]$$

As a quick aside, we can identify the average torque constant, $\hat{k}_t$, for the general case and for the case of a sinusoidal motor. The average motor constant is supposed to relate the trapezoidal current to the average motor torque. Using that definition, we can write:

$$\hat{k}_{t,\text{sine}} = \frac{3\sqrt{3}k_t}{\pi} \quad [1.37]$$

for the case of a sinusoidal motor, and:

$$\hat{k}_t = \frac{3\sqrt{3}k_t}{\pi} \left(1 - \frac{a_5}{5} - \frac{a_7}{7} + \frac{a_{11}}{11} + \frac{a_{13}}{13} - \frac{a_{17}}{17} - \frac{a_{19}}{19} + \dots \right) \quad [1.38]$$

for the more general case. We can also go the other direction by solving for k_t in terms of the average torque constant and the shape of the winding curves (i.e. the values of a_n):

$$k_t = \frac{\pi \hat{k}_t}{3\sqrt{3}} \cdot \frac{1}{\left(1 - \frac{a_5}{5} - \frac{a_7}{7} + \frac{a_{11}}{11} + \frac{a_{13}}{13} - \frac{a_{17}}{17} - \frac{a_{19}}{19} + \dots \right)} \quad [1.39]$$

Derivation of Average Motor Torque using True Sinusoidal Commutation

True sinusoidal commutation implies the motor controller forces the winding currents to be of the form:

$$i_A = I \sin(\theta) \quad [1.40]$$

$$i_B = I \sin(\theta + 120^\circ) \quad [1.41]$$

$$i_C = I \sin(\theta + 240^\circ) \quad [1.42]$$

The average torque produced over one motor revolution is then:

$$\begin{aligned} \hat{T} &= \frac{1}{2\pi} \int_0^{2\pi} k_t I \sin(\theta) \cdot p(\theta) + k_t I \sin(\theta + 120^\circ) \cdot p(\theta + 120^\circ) + k_t I \sin(\theta + 240^\circ) \cdot p(\theta + 240^\circ) d\theta \\ &= \frac{k_t I}{2\pi} \int_0^{2\pi} \sin(\theta) \cdot p(\theta) + \sin(\theta + 120^\circ) \cdot p(\theta + 120^\circ) + \sin(\theta + 240^\circ) \cdot p(\theta + 240^\circ) d\theta \end{aligned} \quad [1.43]$$

Substituting [1.27] in for $p(\theta)$, this becomes:

$$\begin{aligned}
\hat{T} &= \frac{k_t I}{2\pi} \int_0^{2\pi} \left\{ \sin(\theta) \left[\sin(\theta) + \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta) \right] + \right. \\
&\quad \left. \sin(\theta + 120^\circ) \left[\sin(\theta + 120^\circ) + \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) \right] + \right. \\
&\quad \left. \sin(\theta + 240^\circ) \left[\sin(\theta + 240^\circ) + \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) \right] \right\} d\theta \\
&= \frac{k_t I}{2\pi} \int_0^{2\pi} \left\{ \sin^2(\theta) + \sin^2(\theta + 120^\circ) + \sin^2(\theta + 240^\circ) + \right. \\
&\quad \left. \sin(\theta) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta) + \sin(\theta + 120^\circ) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + \sin(\theta + 240^\circ) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) \right\} d\theta \quad [1.44] \\
&= \frac{k_t I}{2\pi} \int_0^{2\pi} \frac{3}{2} + \sin(\theta) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta) + \sin(\theta + 120^\circ) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + \sin(\theta + 240^\circ) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) d\theta \\
&= \frac{3k_t I}{2} + \frac{k_t I}{2\pi} \int_0^{2\pi} \sin(\theta) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta) + \sin(\theta + 120^\circ) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + \sin(\theta + 240^\circ) \sum_{n=3,5,\dots}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) d\theta \\
&= \frac{3k_t I}{2} + \frac{k_t I}{2\pi} \sum_{n=3,5,\dots}^{\infty} \left[a_n \int_0^{2\pi} \sin(\theta) \sin(n\theta) + \sin\left(\theta + \frac{2\pi}{3}\right) \sin\left(n\theta + n \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right) \sin\left(n\theta + n \frac{4\pi}{3}\right) d\theta \right]
\end{aligned}$$

where, among other things, we've used the identity:

$$\sin^2(\theta) + \sin^2(\theta + 120^\circ) + \sin^2(\theta + 240^\circ) = \frac{3}{2} \quad [1.45]$$

Rewriting the last line and rearranging the arguments to the sine functions:

$$\hat{T} = \frac{3k_t I}{2} + \frac{k_t I}{2\pi} \sum_{n=3,5,\dots}^{\infty} \left[a_n \int_0^{2\pi} \sin(\theta) \sin(n\theta) + \sin\left(\theta + \frac{2\pi}{3}\right) \sin\left(n\left(\theta + \frac{2\pi}{3}\right)\right) + \sin\left(\theta + \frac{4\pi}{3}\right) \sin\left(n\left(\theta + \frac{4\pi}{3}\right)\right) d\theta \right] \quad [1.46]$$

We can use the orthogonality property of the sine function:

$$\int_{\theta_0}^{\theta_0 + 2\pi} \sin(m\theta) \sin(n\theta) d\theta = 0 \quad ; \quad \text{for } n \neq m, \quad [1.47]$$

to show that the integral evaluates to zero for each value of n in the summation. This means the average torque is simply given by:

$$\boxed{\hat{T} = \frac{3k_t I}{2}} \quad [1.48]$$

The obvious thing to note here is the non-fundamental components ($a_3, a_5, \dots$) of $p(\theta)$ do not contribute to the average torque produced by the motor.

Derivation of Average Motor Torque using Pseudo-Sinusoidal Commutation

The MSL motor controller does not implement true sinusoidal current commutation, in that it does not force the winding currents to be sinusoidal. Instead, it drives the motor with sinusoidal voltages that would result in sinusoidal winding currents in the case of a truly sinusoidal motor. In the case of a non-sinusoidal motor, the actual winding currents are distorted.

First, let's derive the voltages used to drive the motor. (We know they will be sinusoidal, but their phases are not obvious.) Performing KVL from the A phase to the B phase:

$$V_A - V_B = i_A R + \varepsilon_A - \varepsilon_B - i_B R \quad [1.49]$$

Assuming a sinusoidal motor, the winding EMF's are given by [1.1] through [1.3]. And since we are trying to solve for the voltages needed to make the winding currents sinusoidal, we assume sinusoidal forms for them:

$$\begin{aligned}
V_A - V_B &= I \sin(\theta) R + k_v \dot{\theta} \sin(\theta) - k_v \dot{\theta} \sin(\theta + 120^\circ) - I \sin(\theta + 120^\circ) R \\
&= RI \left[\sin(\theta) - \sin(120^\circ) \right] + k_v \dot{\theta} \left[\sin(\theta) - \sin(\theta + 120^\circ) \right] \\
&= (RI + k_v \dot{\theta}) \left[\sin(\theta) - \sin(120^\circ) \right]
\end{aligned} \tag{1.50}$$

Noting that:

$$\sin(\theta) - \sin(\theta + 120^\circ) = \sqrt{3} \sin(\theta - 30^\circ) \tag{1.51}$$

We can write:

$$V_A - V_B = \sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta - 30^\circ) \tag{1.52}$$

Similarly, we find that:

$$V_A - V_C = \sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta + 30^\circ) \tag{1.53}$$

We can arbitrarily decide to let V_A be our ground reference:

$$V_{A,\text{line}} = 0, \tag{1.54}$$

which means:

$$V_{B,\text{line}} = -\sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta - 30^\circ) \tag{1.55}$$

$$V_{C,\text{line}} = -\sqrt{3} (RI + k_v \dot{\theta}) \sin(\theta + 30^\circ) \tag{1.56}$$

The amplitude of these sinusoids represents the amplitude required to drive a particular sinusoidal motor at a given speed. In practice, the drive voltage amplitude is decided by the motor controller (either specified by the user, or by some other control loop). Therefore, we'll rewrite the voltages as simply:

$$V_{B,\text{line}} = -\sqrt{3} V_0 \sin(\theta - 30^\circ) \tag{1.57}$$

$$V_{C,\text{line}} = -\sqrt{3} V_0 \sin(\theta + 30^\circ) \tag{1.58}$$

The reason for keeping the $\sqrt{3}$ will become apparent down the road.

With the sine drive voltages now known (specifically their phases), we turn back to a non-sinusoidal motor. We want to determine what the winding currents will look like, since we know they will not be purely sinusoidal. Rewriting [1.49], we get:

$$V_{B,\text{line}} = \mathcal{E}_B - \mathcal{E}_A + R(i_B - i_A) \Rightarrow i_B - i_A = \frac{V_{B,\text{line}} - \mathcal{E}_B + \mathcal{E}_A}{R} \tag{1.59}$$

Doing the same (KVL) from phase C to A:

$$V_{C,\text{line}} = \mathcal{E}_C - \mathcal{E}_A + R(i_C - i_A) \Rightarrow i_C - i_A = \frac{V_{C,\text{line}} - \mathcal{E}_C + \mathcal{E}_A}{R} \tag{1.60}$$

These two relations along with the obvious conservation of current:

$$i_A + i_B + i_C = 0 \tag{1.61}$$

yield a system of three equations in three unknowns (the three winding currents). The solution to the system is:

$$i_A = \frac{-V_{B,\text{line}} - V_{C,\text{line}} - 2\mathcal{E}_A + \mathcal{E}_B + \mathcal{E}_C}{3R} \tag{1.62}$$

$$i_B = \frac{2V_{B,\text{line}} - V_{C,\text{line}} + \mathcal{E}_A - 2\mathcal{E}_B + \mathcal{E}_C}{3R} \tag{1.63}$$

$$i_C = \frac{-V_{B,\text{sine}} + 2V_{C,\text{sine}} + \varepsilon_A + \varepsilon_B - 2\varepsilon_C}{3R} \quad [1.64]$$

We now have the winding currents in terms of the drive voltages (given by [1.57] and [1.58]) and the winding EMF's (given in general by [1.7] through [1.9]). Plugging them into [1.10] through [1.12] yields the total motor torque. The average motor torque over one motor revolution is given by:

$$\begin{aligned} \hat{T} &= \frac{1}{2\pi} \int_0^{2\pi} k_t i_A \cdot p(\theta) + k_t i_B \cdot p(\theta + 120^\circ) + k_t i_C \cdot p(\theta + 240^\circ) d\theta \\ &= \frac{k_t}{6\pi R} \int_0^{2\pi} \left[\begin{aligned} &(-V_{B,\text{sine}} - V_{C,\text{sine}} - 2\varepsilon_A + \varepsilon_B + \varepsilon_C) \cdot p(\theta) + \\ &(2V_{B,\text{sine}} - V_{C,\text{sine}} + \varepsilon_A - 2\varepsilon_B + \varepsilon_C) \cdot p(\theta + 120^\circ) + \\ &(-V_{B,\text{sine}} + 2V_{C,\text{sine}} + \varepsilon_A + \varepsilon_B - 2\varepsilon_C) \cdot p(\theta + 240^\circ) \end{aligned} \right] d\theta \end{aligned} \quad [1.65]$$

Plugging in for the drive voltages and winding EMF's:

$$\hat{T} = \frac{k_t}{6\pi R} \int_0^{2\pi} \left\{ \begin{aligned} &\left(\sqrt{3}V_0 [\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ)] + k_v \dot{\theta} [-2 \cdot p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ)] \right) \cdot p(\theta) + \\ &\left(\sqrt{3}V_0 [-2\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ)] + k_v \dot{\theta} [p(\theta) - 2 \cdot p(\theta + 120^\circ) + p(\theta + 240^\circ)] \right) \cdot p(\theta + 120^\circ) + \\ &\left(\sqrt{3}V_0 [\sin(\theta - 30^\circ) - 2\sin(\theta + 30^\circ)] + k_v \dot{\theta} [p(\theta) + p(\theta + 120^\circ) - 2 \cdot p(\theta + 240^\circ)] \right) \cdot p(\theta + 240^\circ) \end{aligned} \right\} d\theta \quad [1.66]$$

We can make use of the following identities:

$$\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ) = \sqrt{3} \sin(\theta) \quad [1.67]$$

$$-2\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ) = \sqrt{3} \sin(\theta + 120^\circ) \quad [1.68]$$

$$\sin(\theta - 30^\circ) - 2\sin(\theta + 30^\circ) = \sqrt{3} \sin(\theta + 240^\circ) \quad [1.69]$$

to simplify [1.66] to:

$$\hat{T} = \frac{k_t}{6\pi R} \int_0^{2\pi} \left\{ \begin{aligned} &\left(3V_0 \sin(\theta) + k_v \dot{\theta} [-2 \cdot p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ)] \right) \cdot p(\theta) + \\ &\left(3V_0 \sin(\theta + 120^\circ) + k_v \dot{\theta} [p(\theta) - 2 \cdot p(\theta + 120^\circ) + p(\theta + 240^\circ)] \right) \cdot p(\theta + 120^\circ) + \\ &\left(3V_0 \sin(\theta + 240^\circ) + k_v \dot{\theta} [p(\theta) + p(\theta + 120^\circ) - 2 \cdot p(\theta + 240^\circ)] \right) \cdot p(\theta + 240^\circ) \end{aligned} \right\} d\theta \quad [1.70]$$

Now expanding and utilizing orthogonality to eliminate some terms, we get:

$$\begin{aligned} \hat{T} &= \frac{k_t}{6\pi R} \int_0^{2\pi} \left\{ \begin{aligned} &\left(3V_0 \sin(\theta) \cdot p(\theta) + k_v \dot{\theta} [-2 \cdot p(\theta)^2 + p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ)] \right) + \\ &\left(3V_0 \sin(\theta + 120^\circ) \cdot p(\theta + 120^\circ) + k_v \dot{\theta} [p(\theta) \cdot p(\theta + 120^\circ) - 2 \cdot p(\theta + 120^\circ)^2 + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ)] \right) + \\ &\left(3V_0 \sin(\theta + 240^\circ) \cdot p(\theta + 240^\circ) + k_v \dot{\theta} [p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) - 2 \cdot p(\theta + 240^\circ)^2] \right) \end{aligned} \right\} d\theta \\ &= \frac{k_t}{6\pi R} \int_0^{2\pi} \left\{ \begin{aligned} &\left(3V_0 a_1 \sin^2(\theta) + k_v \dot{\theta} [-2 \cdot p(\theta)^2 + p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ)] \right) + \\ &\left(3V_0 a_1 \sin^2(\theta + 120^\circ) + k_v \dot{\theta} [p(\theta) \cdot p(\theta + 120^\circ) - 2 \cdot p(\theta + 120^\circ)^2 + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ)] \right) + \\ &\left(3V_0 a_1 \sin^2(\theta + 240^\circ) + k_v \dot{\theta} [p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) - 2 \cdot p(\theta + 240^\circ)^2] \right) \end{aligned} \right\} d\theta \end{aligned} \quad [1.71]$$

Collecting terms:

$$\begin{aligned}
\hat{T} &= \frac{k_t}{6\pi R} \int_0^{2\pi} \left\{ 3V_0 a_1 \left[\sin^2(\theta) + \sin^2(\theta + 120^\circ) + \sin^2(\theta + 240^\circ) \right] + \right. \\
&\quad \left. k_v \dot{\theta} \left[\begin{aligned} & \left[-2 \cdot p(\theta)^2 + p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 120^\circ) - 2 \cdot p(\theta + 120^\circ)^2 + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) - 2 \cdot p(\theta + 240^\circ)^2 \right] \end{aligned} \right] \right\} d\theta \\
&= \frac{k_t}{6\pi R} \int_0^{2\pi} \left\{ \frac{9}{2} V_0 a_1 + k_v \dot{\theta} \left[\begin{aligned} & \left[-2 \cdot p(\theta)^2 + p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 120^\circ) - 2 \cdot p(\theta + 120^\circ)^2 + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) - 2 \cdot p(\theta + 240^\circ)^2 \right] \end{aligned} \right] \right\} d\theta
\end{aligned} \tag{1.72}$$

Separating into two integrals and assuming our $p(\theta)$ is normalized (constraint [1.26]):

$$\begin{aligned}
\hat{T} &= \frac{k_t}{6\pi R} \int_0^{2\pi} \frac{9}{2} V_0 d\theta + \frac{k_t}{6\pi R} \int_0^{2\pi} k_v \dot{\theta} \left\{ \begin{aligned} & \left[-2 \cdot p(\theta)^2 + p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 120^\circ) - 2 \cdot p(\theta + 120^\circ)^2 + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) - 2 \cdot p(\theta + 240^\circ)^2 \right] \end{aligned} \right\} d\theta \\
&= \frac{3k_t V_0}{2R} + \frac{k_t k_v \dot{\theta}}{6\pi R} \int_0^{2\pi} \left\{ \begin{aligned} & \left[-2 \cdot p(\theta)^2 + p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 120^\circ) - 2 \cdot p(\theta + 120^\circ)^2 + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] + \\ & \left[p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) - 2 \cdot p(\theta + 240^\circ)^2 \right] \end{aligned} \right\} d\theta \\
&= \frac{3k_t V_0}{2R} + \frac{k_t k_v \dot{\theta}}{6\pi R} \int_0^{2\pi} \left\{ \begin{aligned} & -2 \cdot p(\theta)^2 - 2 \cdot p(\theta + 120^\circ)^2 - 2 \cdot p(\theta + 240^\circ)^2 + \\ & 2 \cdot p(\theta) \cdot p(\theta + 120^\circ) + 2 \cdot p(\theta) \cdot p(\theta + 240^\circ) + 2 \cdot p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \end{aligned} \right\} d\theta \\
&= \frac{3k_t V_0}{2R} + \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left\{ \begin{aligned} & - \left[p(\theta)^2 + p(\theta + 120^\circ)^2 + p(\theta + 240^\circ)^2 \right] + \\ & p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \end{aligned} \right\} d\theta
\end{aligned} \tag{1.73}$$

Using orthogonality to simplify the bracketed term in the integral:

$$\begin{aligned}
\hat{T} &= \frac{3k_t V_0}{2R} + \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left\{ - \left[\sum_{n=1}^{\infty} a_n^2 \sin^2(n\theta) + \sum_{n=1}^{\infty} a_n^2 \sin^2(n(\theta + 120^\circ)) + \sum_{n=1}^{\infty} a_n^2 \sin^2(n(\theta + 240^\circ)) \right] + \right. \\
&\quad \left. p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right\} d\theta \\
&= \left[\begin{aligned} & \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \sum_{n=1}^{\infty} a_n^2 \left[\sin^2(n\theta) + \sin^2(n(\theta + 120^\circ)) + \sin^2(n(\theta + 240^\circ)) \right] d\theta + \\ & \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left[p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] d\theta \end{aligned} \right] \\
&= \left[\begin{aligned} & \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{3\pi R} \sum_{n=1}^{\infty} \left\{ a_n^2 \int_0^{2\pi} \left[\sin^2(n\theta) + \sin^2(n(\theta + 120^\circ)) + \sin^2(n(\theta + 240^\circ)) \right] d\theta \right\} + \\ & \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left[p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] d\theta \end{aligned} \right]
\end{aligned} \tag{1.74}$$

Noting that:

$$\int_0^{2\pi} \sin^2(ax + b) dx = \pi \tag{1.75}$$

Now [1.74] simplifies to:

$$\begin{aligned}\hat{T} &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{3\pi R} \sum_{n=1}^{\infty} a_n^2 3\pi + \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left[p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] d\theta \quad [1.76] \\ &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left[p(\theta) \cdot p(\theta + 120^\circ) + p(\theta) \cdot p(\theta + 240^\circ) + p(\theta + 120^\circ) \cdot p(\theta + 240^\circ) \right] d\theta\end{aligned}$$

Now we attack the last integral. Just as before, orthogonality gets rid of all the cross terms:

$$\begin{aligned}\hat{T} &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} a_n^2 \sin(n\theta) \sin(n\theta + n \cdot 120^\circ) + \sum_{n=1}^{\infty} a_n^2 \sin(n\theta) \sin(n\theta + n \cdot 240^\circ) + \sum_{n=1}^{\infty} a_n^2 \sin(n\theta + n \cdot 120^\circ) \sin(n\theta + n \cdot 240^\circ) \right] d\theta \quad [1.77] \\ &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3\pi R} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} a_n^2 \left[\sin(n\theta) \sin(n\theta + n \cdot 120^\circ) + \sin(n\theta) \sin(n\theta + n \cdot 240^\circ) + \sin(n\theta + n \cdot 120^\circ) \sin(n\theta + n \cdot 240^\circ) \right] \right] d\theta \\ &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3\pi R} \sum_{n=1}^{\infty} a_n^2 \left\{ \int_0^{2\pi} \left[\sin(n\theta) \sin(n\theta + n \cdot 120^\circ) + \sin(n\theta) \sin(n\theta + n \cdot 240^\circ) + \sin(n\theta + n \cdot 120^\circ) \sin(n\theta + n \cdot 240^\circ) \right] d\theta \right\} \neq\end{aligned}$$

Noting that:

$$\sin(a + b) = \cos(b) \sin(a) + \sin(b) \cos(a) \quad [1.78]$$

and that (more orthogonality):

$$\int_0^{2\pi} \sin(ax) \cos(ax) dx = 0 \quad [1.79]$$

We can simplify [1.77] to:

$$\begin{aligned}\hat{T} &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3\pi R} \sum_{n=1}^{\infty} a_n^2 \left\{ \int_0^{2\pi} \left[\cos(n \cdot 120^\circ) \sin^2(n\theta) + \cos(n \cdot 240^\circ) \sin^2(n\theta) + \cos(n \cdot 120^\circ) \sin^2(n\theta + n \cdot 120^\circ) \right] d\theta \right\} \quad [1.80] \\ &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3\pi R} \sum_{n=1}^{\infty} a_n^2 \pi \left[\cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ) + \cos(n \cdot 120^\circ) \right] \\ &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3R} \sum_{n=1}^{\infty} a_n^2 \left[2 \cos(n \cdot 120^\circ) + \cos(n \cdot 240^\circ) \right] \\ &= \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} \sum_{n=1}^{\infty} a_n^2 + \frac{k_t k_v \dot{\theta}}{3R} \sum_{n=1}^{\infty} a_n^2 \left[2 \cos\left(n \cdot \frac{2\pi}{3}\right) + \cos\left(n \cdot \frac{4\pi}{3}\right) \right]\end{aligned}$$

Expanding the summations:

$$\begin{aligned}\hat{T} &= \left\{ \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \dots) + \frac{k_t k_v \dot{\theta}}{3R} \left(-\frac{3}{2} a_1^2 - \frac{3}{2} a_2^2 + 3a_3^2 - \frac{3}{2} a_4^2 - \frac{3}{2} a_5^2 + 3a_6^2 - \frac{3}{2} a_7^2 \dots \right) \right\} = \left\{ \frac{3k_t V_0}{2R} - \frac{k_t k_v \dot{\theta}}{R} (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \dots) + \frac{k_t k_v \dot{\theta}}{R} \left(-\frac{1}{2} a_1^2 - \frac{1}{2} a_2^2 + a_3^2 - \frac{1}{2} a_4^2 - \frac{1}{2} a_5^2 + a_6^2 - \frac{1}{2} a_7^2 \dots \right) \right\} \quad [1.81] \\ &= \frac{3k_t V_0}{2R} + \frac{k_t k_v \dot{\theta}}{R} \left[-(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \dots) + \left(-\frac{1}{2} a_1^2 - \frac{1}{2} a_2^2 + a_3^2 - \frac{1}{2} a_4^2 - \frac{1}{2} a_5^2 + a_6^2 - \frac{1}{2} a_7^2 \dots \right) \right] \\ &= \frac{3k_t V_0}{2R} + \frac{k_t k_v \dot{\theta}}{R} \left(-\frac{3}{2} a_1^2 - \frac{3}{2} a_2^2 - \frac{3}{2} a_4^2 - \frac{3}{2} a_5^2 - \frac{3}{2} a_7^2 - \frac{3}{2} a_8^2 - \frac{3}{2} a_{10}^2 - \frac{3}{2} a_{11}^2 \dots \right) \\ &= \frac{3k_t V_0}{2R} + \frac{3k_t k_v \dot{\theta}}{2R} (-a_1^2 - a_2^2 - a_4^2 - a_5^2 - a_7^2 - a_8^2 - a_{10}^2 - a_{11}^2 \dots) \\ &= \frac{3k_t V_0}{2R} - \frac{3k_t k_v \dot{\theta}}{2R} (a_1^2 + a_2^2 + a_4^2 + a_5^2 + a_7^2 + a_8^2 + a_{10}^2 + a_{11}^2 + \dots)\end{aligned}$$

And again assuming a normalized $p(\theta)$, this becomes:

$$\hat{T} = \frac{3k_t}{2} \left(\frac{V_0 - k_v \dot{\theta}}{R} \right) - \frac{3k_t k_v \dot{\theta}}{2R} (a_2^2 + a_4^2 + a_5^2 + a_7^2 + a_8^2 + a_{10}^2 + a_{11}^2 + \dots) \quad [1.82]$$

Note that the first term on the right represents the average torque that would be produced if true sinusoidal commutation was being used (with $(V_0 - k_v \dot{\theta})/R$ in place of I). And as we'd expect (since this derivation was based on the premise), [1.82] does reduce to this when $p(\theta)$ is a sinusoid (i.e. when $a_n = 0$; for $n = 2, 3, \dots$). We can also see that the impact of $p(\theta)$ having non-zero non-fundamental components is always to decrease the amount of average torque produced (since the second term in [1.82] is always positive). Furthermore, we see that for a given non-sinusoidal motor,

the torque deviation between pseudo-sinusoidal commutation and true sinusoidal commutation is proportional to motor speed.

Assuming the symmetry constraint in [1.24] is also satisfied, [1.82] reduces to:

$$\hat{T} = \frac{3k_t}{2} \left(\frac{V_0 - k_v \dot{\theta}}{R} \right) - \frac{3k_t k_v \dot{\theta}}{2R} (a_5^2 + a_7^2 + a_{11}^2 + a_{13}^2 + a_{17}^2 + a_{19}^2 + \dots) \quad [1.83]$$

Interestingly, the same harmonics are involved as in equation [1.35]. We can rearrange this to the following form:

$$\hat{T} = \frac{3k_t}{2} \left[\frac{V_0 - k_v \dot{\theta}}{R} - \frac{k_v \dot{\theta}}{R} (a_5^2 + a_7^2 + a_{11}^2 + a_{13}^2 + a_{17}^2 + a_{19}^2 + \dots) \right] \quad [1.84]$$

Determining Motor Torque from Experimental Winding Current Data

Assume that a non-sinusoidal motor is being driven by a motor controller employing an unidentified commutation scheme. All that is available is experimental winding current data. This analysis will assume the winding currents are of the form:

$$i_A = q(\theta) \quad [1.85]$$

$$i_B = q(\theta + 120^\circ) \quad [1.86]$$

$$i_C = q(\theta + 240^\circ) \quad [1.87]$$

where,

$$q(\theta) = \sum_{n=1}^{\infty} c_n \sin(n\theta) + d_n \cos(n\theta) \quad [1.88]$$

The implicit assumption built into these equations is that each of the three winding currents look exactly the same (in shape and amplitude), but are just shifted by 120 degrees from one another. This should correspond to symmetry in the motor's stator. Using the above expressions for current, the average torque produced over one motor revolution is given by:

$$\begin{aligned} \hat{T} &= \frac{1}{2\pi} \int_0^{2\pi} k_t i_A \cdot p(\theta) + k_t i_B \cdot p(\theta + 120^\circ) + k_t i_C \cdot p(\theta + 240^\circ) d\theta \\ &= \frac{k_t}{2\pi} \int_0^{2\pi} q(\theta) \cdot p(\theta) + q(\theta + 120^\circ) \cdot p(\theta + 120^\circ) + q(\theta + 240^\circ) \cdot p(\theta + 240^\circ) d\theta \end{aligned} \quad [1.89]$$

Assuming $p(\theta)$ is of the form in [1.25]:

$$\hat{T} = \frac{k_t}{2\pi} \int_0^{2\pi} \left[\sum_{n=1}^{\infty} c_n \sin(n\theta) + d_n \cos(n\theta) \cdot \sum_{n=1,3,\dots}^{\infty} a_n \sin(n\theta) + \right. \\ \left. \sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 120^\circ) + d_n \cos(n\theta + n \cdot 120^\circ) \cdot \sum_{n=1,3,\dots}^{\infty} a_n \sin(n\theta + n \cdot 120^\circ) + \right. \\ \left. \sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 240^\circ) + d_n \cos(n\theta + n \cdot 240^\circ) \cdot \sum_{n=1,3,\dots}^{\infty} a_n \sin(n\theta + n \cdot 240^\circ) \right] d\theta \quad [1.90]$$

Again using orthogonality and a bunch of other tricks to simplify, this becomes:

$$\begin{aligned}
\hat{T} &= \frac{k_t}{2\pi} \int_0^{2\pi} \left[\sum_{n=1,3,\dots}^{\infty} c_n a_n \sin^2(n\theta) + \sum_{n=1,3,\dots}^{\infty} c_n a_n \sin^2(n\theta + n \cdot 120^\circ) + \sum_{n=1,3,\dots}^{\infty} c_n a_n \sin^2(n\theta + n \cdot 240^\circ) \right] d\theta \\
&= \frac{k_t}{2\pi} \int_0^{2\pi} \sum_{n=1,3,\dots}^{\infty} c_n a_n \left[\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ) \right] d\theta \\
&= \frac{k_t}{2\pi} \sum_{n=1,3,\dots}^{\infty} \left\{ c_n a_n \int_0^{2\pi} \left[\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ) \right] d\theta \right\} \\
&= \frac{k_t}{2\pi} \sum_{n=1,3,\dots}^{\infty} 3\pi c_n a_n \\
&= \frac{3k_t}{2} \sum_{n=1,3,\dots}^{\infty} c_n a_n
\end{aligned} \tag{1.91}$$

Assuming that $p(\theta)$ is normalized, this becomes:

$$\hat{T} = \frac{3k_t}{2} (c_1 + c_3 a_3 + c_5 a_5 + c_7 a_7 + \dots) \tag{1.92}$$

A huge amount of information is contained in this equation:

1. If the motor is a sinusoidal motor ($a_n = 0$; for $n = 2, 3, 4, \dots$), then the average torque produced by the motor is simply proportional to the first Fourier sine coefficient of the current waveform. The type of commutation is irrelevant, as long as it satisfies the loose assumption described above. This even works for trapezoidal commutation!
2. If the winding currents are sinusoidal ($c_n = 0$; for $n = 2, 3, 4, \dots$), this equation reduces to the familiar equation for sinusoidal commutation (with $I = c_1$). This is no surprise, but it took less work to show it this way. In fact, the winding currents don't need to be purely sinusoidal for this to apply. As seen in [1.92], the currents can have distortion at any or all of the even Fourier components, and the average torque will still just be proportional to the fundamental component's amplitude, c_1 .
3. In the case of a non-sinusoidal motor and a non-sinusoidal winding current, the average torque produced can be more than, equal to, or less than what would be predicted by only accounting for the first (fundamental) Fourier coefficient.

Now substituting [1.39] into [1.91] for k_t :

$$\hat{T} = \frac{\pi \hat{k}_t}{2\sqrt{3}} \cdot \frac{c_1 + c_3 a_3 + c_5 a_5 + c_7 a_7 + \dots}{1 - \frac{a_5}{5} - \frac{a_7}{7} + \frac{a_{11}}{11} + \frac{a_{13}}{13} - \frac{a_{17}}{17} - \frac{a_{19}}{19} + \dots} \tag{1.93}$$

If average torque was also measured and the objective was to determine the average torque constant, we can solve for $\hat{k}_t$:

$$\hat{k}_t = \frac{2\sqrt{3}}{\pi} \cdot \frac{1 - \frac{a_5}{5} - \frac{a_7}{7} + \frac{a_{11}}{11} + \frac{a_{13}}{13} - \frac{a_{17}}{17} - \frac{a_{19}}{19} + \dots}{c_1 + c_3 a_3 + c_5 a_5 + c_7 a_7 + \dots} \cdot \hat{T} \tag{1.94}$$

A quick sanity check on this is to consider the case of a sinusoidal motor (i.e. $a_n = 0$; for $n = 2, 3, \dots$):

$$\hat{k}_{t,\text{ sine}} = \frac{2\sqrt{3}}{\pi} \cdot \frac{\hat{T}}{c_1} \tag{1.95}$$

where recall that c_1 is simply the first Fourier sine coefficient of winding current (i.e. the fundamental component of current in phase with the winding curve, $p(\theta)$).

Deriving Winding Dissipation from Experimental Winding Current Data

If winding current data is available, the average power dissipation in the winding resistance is given by:

$$\begin{aligned}
 \hat{P}_{diss,w} &= \frac{1}{2\pi} \int_0^{2\pi} R(i_A^2 + i_B^2 + i_C^2) d\theta \\
 &= \frac{R}{2\pi} \int_0^{2\pi} q(\theta)^2 + q(\theta + 120^\circ)^2 + q(\theta + 240^\circ)^2 d\theta \\
 &= \frac{R}{2\pi} \int_0^{2\pi} \left\{ \left[\sum_{n=1}^{\infty} c_n \sin(n\theta) + d_n \cos(n\theta) \right]^2 + \right. \\
 &\quad \left. \left[\sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 120^\circ) + d_n \cos(n\theta + n \cdot 120^\circ) \right]^2 + \right. \\
 &\quad \left. \left[\sum_{n=1}^{\infty} c_n \sin(n\theta + n \cdot 240^\circ) + d_n \cos(n\theta + n \cdot 240^\circ) \right]^2 \right\} d\theta
 \end{aligned} \tag{1.96}$$

Again we use orthogonality to simplify this to:

$$\begin{aligned}
 \hat{P}_{diss,w} &= \frac{R}{2\pi} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} c_n^2 \sin^2(n\theta) + d_n^2 \cos^2(n\theta) + \right. \\
 &\quad \left. \sum_{n=1}^{\infty} c_n^2 \sin^2(n\theta + n \cdot 120^\circ) + d_n^2 \cos^2(n\theta + n \cdot 120^\circ) + \right. \\
 &\quad \left. \sum_{n=1}^{\infty} c_n^2 \sin^2(n\theta + n \cdot 240^\circ) + d_n^2 \cos^2(n\theta + n \cdot 240^\circ) \right\} d\theta \\
 &= \frac{R}{2\pi} \int_0^{2\pi} \left\{ \sum_{n=1}^{\infty} c_n^2 \left[\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ) \right] + \right. \\
 &\quad \left. \sum_{n=1}^{\infty} d_n^2 \left[\cos^2(n\theta) + \cos^2(n\theta + n \cdot 120^\circ) + \cos^2(n\theta + n \cdot 240^\circ) \right] \right\} d\theta \\
 &= \frac{R}{2\pi} \sum_{n=1}^{\infty} \left\{ c_n^2 \int_0^{2\pi} \left[\sin^2(n\theta) + \sin^2(n\theta + n \cdot 120^\circ) + \sin^2(n\theta + n \cdot 240^\circ) \right] d\theta + \right. \\
 &\quad \left. d_n^2 \int_0^{2\pi} \left[\cos^2(n\theta) + \cos^2(n\theta + n \cdot 120^\circ) + \cos^2(n\theta + n \cdot 240^\circ) \right] d\theta \right\} \\
 &= \frac{R}{2\pi} \sum_{n=1}^{\infty} (c_n^2 \cdot 3\pi + d_n^2 \cdot 3\pi) \\
 &= \frac{3R}{2} \sum_{n=1}^{\infty} (c_n^2 + d_n^2)
 \end{aligned} \tag{1.97}$$

Rewriting this in its final form:

$$\boxed{\hat{P}_{diss,w} = \frac{3R}{2} \sum_{n=1}^{\infty} (c_n^2 + d_n^2)} \tag{1.98}$$

Recall (as can be shown using similar methods as above) that the sum of the squares of the Fourier coefficients is related to a function's RMS value by:

$$\sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 2 \cdot f_{RMS}^2 \quad ; \quad \text{for } f(\theta) = \sum_{n=1}^{\infty} a_n \sin(n\theta) + b_n \cos(n\theta) \quad [1.99]$$

Rewriting the average dissipation in terms of the RMS winding current:

$$\hat{P}_{diss,w} = 3 \cdot q_{RMS}^2 \cdot R \quad [1.100]$$

Now that I'm here, this should have been obvious – it's the simple Joule heating relationship for AC current (the factor of three is there to account for all three windings).

Estimating Efficiency from Experimental Winding Current, Torque, and Speed Data

The motor efficiency is defined by:

$$\eta \equiv \frac{P_{mech_output}}{P_{input}} \quad [1.101]$$

where P_{mech_output} denotes the mechanical output (shaft) power from the motor, and P_{input} denotes the total electrical power into the motor. All the input power must go somewhere – it can be stored (in the motor), dissipated (as Joule heating, hysteresis loss, viscous drag, etc.) , and/or transmitted as output power:

$$P_{input} = \dot{E}_{motor} + P_{diss} + P_{mech_output} \quad [1.102]$$

What we're really after is average efficiency more so than instantaneous efficiency. we'll define it as (note this is not the same thing as mathematically averaging [1.101]):

$$\hat{\eta} \equiv \frac{\hat{P}_{mech_output}}{\hat{P}_{input}} \quad [1.103]$$

Realizing that the average rate of energy stored in the motor must be zero at steady state, we can plug the average of [1.102] into [1.103] to eliminate $\hat{P}_{input}$ (mainly because it's difficult to measure):

$$\hat{\eta} \equiv \frac{\hat{P}_{mech_output}}{\hat{P}_{diss} + \hat{P}_{mech_output}} \quad [1.104]$$

Now we make an assumption that the average power dissipation is largely dominated by Joule heating in the windings. In other words, we neglect other losses such as hysteresis loss, viscous drag, etc. In practice for most motors, this has shown to be true. In this case, the dissipation is give by equation [1.100].

Note that the mechanical output power is given by:

$$P_{mech_output} = T \cdot \dot{\theta} \Rightarrow \hat{P}_{mech_output} \approx \hat{T} \cdot \hat{\dot{\theta}} \quad [1.105]$$

If torque and speed measurement data is available, the motor efficiency is given by:

$$\hat{\eta} \equiv \frac{\hat{T} \cdot \hat{\dot{\theta}}}{3 \cdot q_{RMS}^2 \cdot R + \hat{T} \cdot \hat{\dot{\theta}}} \quad [1.106]$$

Now we will examine the effect winding curve distortion has on a motor's efficiency while being operated with pseudo-sinusoidal commutation. To do this, we need to determine the RMS winding current.

$$\begin{aligned}
i_{A,RMS} &= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} i_A^2 d\theta} \\
&= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{-V_{B,\text{ sine}} - V_{C,\text{ sine}} - 2\varepsilon_A + \varepsilon_B + \varepsilon_C}{3R} \right)^2 d\theta} \\
&= \sqrt{\frac{1}{18\pi R^2} \int_0^{2\pi} \left(-V_{B,\text{ sine}} - V_{C,\text{ sine}} - 2\varepsilon_A + \varepsilon_B + \varepsilon_C \right)^2 d\theta}
\end{aligned}$$

Leveraging off of...

$$\begin{aligned}
i_{A,RMS} &= \sqrt{\frac{1}{18\pi R^2} \int_0^{2\pi} \left\{ 3V_0 \sin(\theta) + k_v \dot{\theta} \left[-2 \cdot p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ) \right] \right\}^2 d\theta} \\
&= \sqrt{\frac{1}{18\pi R^2} \int_0^{2\pi} \left\{ 9V_0^2 \sin^2(\theta) + k_v^2 \dot{\theta}^2 \left[-2 \cdot p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ) \right]^2 + \right. \\
&\quad \left. 6V_0 \sin(\theta) \cdot k_v \dot{\theta} \left[-2 \cdot p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ) \right] \right\} d\theta}
\end{aligned}$$

$$\begin{aligned}
\hat{T} &= \frac{1}{2\pi} \int_0^{2\pi} k_t i_A \cdot p(\theta) + k_t i_B \cdot p(\theta + 120^\circ) + k_t i_C \cdot p(\theta + 240^\circ) d\theta \\
&= \frac{k_t}{6\pi R} \int_0^{2\pi} \left[\begin{aligned} &(-V_{B,\text{ sine}} - V_{C,\text{ sine}} - 2\varepsilon_A + \varepsilon_B + \varepsilon_C) \cdot p(\theta) + \\ &(2V_{B,\text{ sine}} - V_{C,\text{ sine}} + \varepsilon_A - 2\varepsilon_B + \varepsilon_C) \cdot p(\theta + 120^\circ) + \\ &(-V_{B,\text{ sine}} + 2V_{C,\text{ sine}} + \varepsilon_A + \varepsilon_B - 2\varepsilon_C) \cdot p(\theta + 240^\circ) \end{aligned} \right] d\theta \tag{1.107}
\end{aligned}$$

Plugging in for the drive voltages and winding EMF's:

$$\hat{T} = \frac{k_t}{6\pi R} \int_0^{2\pi} \left\{ \begin{aligned} &(\sqrt{3}V_0 [\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ)] + k_v \dot{\theta} [-2 \cdot p(\theta) + p(\theta + 120^\circ) + p(\theta + 240^\circ)]) \cdot p(\theta) + \\ &(\sqrt{3}V_0 [-2\sin(\theta - 30^\circ) + \sin(\theta + 30^\circ)] + k_v \dot{\theta} [p(\theta) - 2 \cdot p(\theta + 120^\circ) + p(\theta + 240^\circ)]) \cdot p(\theta + 120^\circ) + \\ &(\sqrt{3}V_0 [\sin(\theta - 30^\circ) - 2\sin(\theta + 30^\circ)] + k_v \dot{\theta} [p(\theta) + p(\theta + 120^\circ) - 2 \cdot p(\theta + 240^\circ)]) \cdot p(\theta + 240^\circ) \end{aligned} \right\} d\theta \tag{1.108}$$